Exercises

Sequences and Limits

Exercise 1. Determine if the following sequences converge and compute the limit if it exists.

1. $a_n = \dfrac{4n^2 + 3n - 27}{8n^2 - 24n + 108}$
2. $b_n = \dfrac{5n^3 - 6n}{8n^4 - 3}$
3. $c_n = \dfrac{n^2 - n + 5}{n + 8}$
4. $d_n = \sqrt{n^2 + n + 1} - \sqrt{n^2 + 1}$
   Hint: Multiply the sequence by $\frac{\sqrt{n^2+n+1}+\sqrt{n^2+1}}{\sqrt{n^2+n+1}+\sqrt{n^2+1}}$ and use the continuity of the root function (i.e. $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$).

Exercise 2. The following statements are **not true**! Find counterexamples.

1. If a_n and b_n are divergent, then $a_n + b_n$ is also divergent.
2. If $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = -\infty$ then $\lim\limits_{n\to\infty} (a_n + b_n) = 0$.
3. If a_n is strictly increasing (i.e. $a_{n+1} > a_n$ for all $n \in \mathbb{N}$) then $\lim\limits_{n\to\infty} a_n = \infty$.
4. If $\lim\limits_{n\to\infty} a_n = 0$ then for every sequence $(b_n)_n$ we have $\lim\limits_{n\to\infty} a_n b_n = 0$
5. If a_n is bounded, i.e. there is $M \in R$ such that $|a_n| \leq M$, then a_n is convergent.

Exercise 3. A customer wants to invest an amount C_0 for some fixed annual interest rate of p.

1. Give a sequence a_n such that a_n is the capital after n years.
2. How much money C_0 has to be invested if the customer wants to have 10000 € after 10 years when the interest rate is $p = 5\%$.
3. If the customer invests $C_0 = 10000$ € and the interest rate is $p = 4\%$ what is the capital after 20 years?

4. (a) How many years does it take to double the capital if $p = 3\%$?
   (b) How many years does it take to double the capital if $p = 6\%$?
   (c) How many years does it take to double the capital depending on p?

Exercise 4. A sequence is called a **recursive sequence** if a_{n+1} is given implicitly as a function of $a_0, \dots, a_n$.

1. Consider the **Fibonacci numbers** defined by

$$a_0 = 1, a_1 = 1$$
$$a_{n+1} = a_n + a_{n-1} \quad \text{for } n \geq 1$$

   Compute the Fibonacci numbers $a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}$.

2. Consider the recursive sequence defined by

$$a_0 = 0, \quad a_{n+1} = \frac{1}{2}(a_n + 1)$$

   (a) Prove that $a_n \leq 1$ for all $n \in \mathbb{N}$ via induction.
   (b) Use (a) to show that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, i.e. to show that a_n is increasing.
   (c) (a) and (b) imply that a_n is convergent (Can you give a brief explanation why?). Compute $a = \lim_{n\to\infty} a_n$ by using the recursive definition, the rules for the computation of limits and the fact that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a.$$